Systems of parameters

(Assume all rings are Noetherian)

Using the PIT, we get the following characterization of dimension of a local ring:

Prop: If R is a local ring w/ maximal ideal m, then $\dim R$ is the minimum d s.t. $\exists$ d elements $x_1, \dots, x_d \in m$ with $m^n \subseteq (x_1, \dots, x_d)$ for $n >> 0$. (Or, equivalently, by the cor about local Artinian rings, m is minimal over $(x_1, \dots, x_d)$.)

Pf: By PIT, $\dim R = \operatorname{codim} m \leq d$.

For the other inequality, set $c = \operatorname{codim} m = \dim R$. Then by the PIT converse, we can find

$$I = (x_1, \dots, x_c) \subseteq m$$

with m minimal over I. By minimality of d, we have $d \leq c = \dim R$. $\square$

A sequence of elements as in the prop is called a system of parameters for R. More precisely:

Def: If (R, m) is local of dim d, $x_1, \dots, x_d$ is a system of parameters for R if any of the following equivalent conditions hold:

1.) $\mathfrak{m}$ is minimal over $(x_1, \ldots, x_d)$

2.) $\mathfrak{m}^n \subseteq (x_1, \ldots, x_d)$ for $n >> 0$.

3.) $R/(x_1, \ldots, x_d)$ is Artinian.

4.) $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$ (since $\mathfrak{m}$ is the only max'l ideal)

5.) $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary (since the only prime in $R/(x_1, \ldots, x_d)$ is $\mathfrak{m}$, so it's the only ass. prime)

Regular local rings

Let $(R, \mathfrak{m})$ be a local ring of dimension d. Then $\mathfrak{m}$ can't be generated by fewer than d elements.

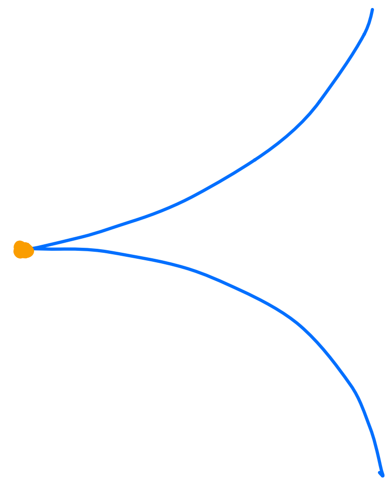
$\mathfrak{m}$ is generated by exactly d elements $\Leftrightarrow$ it's generated by a system of parameters. Such rings are called <u>regular local rings</u>.

Note that not all local rings are regular:

Ex: $R = k[x]/(x^2)$. $\operatorname{Spec} R = (x)$, so R is local and $\dim R = 0$. Thus, x is not a s.o.p. for $\mathfrak{m} = (x)$, so R

is not regular.

Ex: Let $R = \left(\mathbb{C}[x,y]/(y^2-x^3)\right)_{(x,y)}$.



$m=(x,y)$ is not principal in R,

but $(x,y)^2 = (x^2, xy, y^2) = (x^2, xy, x^3) \subseteq (x)$

Thus, $\dim R \le 1$, but 0 is prime so $\dim R = 1$. Thus R is not regular, and x is a system of parameters.

However, if you localize at any other max'l ideal, it will be regular (check!)

More generally, a point on a variety (or scheme) is smooth $\iff$ the corresponding local ring is regular.

Prop: Regular local rings are integral domains.

Proof is tricky — see e.g. A-M.

Systems of parameters are related to the following:

Def: A sequence $x_1, \dots, x_d$ of elements in a ring R is an <u>R-sequence</u> or <u>regular sequence</u> if $(x_1, \dots, x_d) \subsetneq R$ and for each i, x_{i+1} is a NZD on $R/(x_1, \dots, x_i)$.

In general, order matters:

Ex: In $\mathbb{C}[x,y,z]$, $x, y(1-x), z(1-x)$ is a regular sequence:

$$\overline{y(1-x)} = \overline{y} \quad \text{in} \quad \mathbb{C}[x,y,z]/(x)$$

and $(x, y(1-x)) = (x,y)$, so

$$\overline{z(1-x)} = \overline{z} \quad \text{in} \quad \mathbb{C}[x,y,z]/(x,y)$$

However, $y(1-x), z(1-x), x$ is not a regular sequence: $z(1-x)$ is a zero divisor on $\mathbb{C}[x,y,z]/(y(1-x))$.

In special situations, order doesn't matter:

Cor: If $x_1, \dots, x_d$ is a system of parameters in a regular local ring, then $x_1, \dots, x_d$ is a regular sequence.

Pf: For each i, the ring $R/(x_1, \dots, x_i)$ is local of dimension $\geq d-i$. Why? This follows from the following fact about f.g. modules over local rings:

Claim: If $(R, \mathfrak{m})$ is local and M a finitely generated

R-module, then for any $x \in \mathfrak{m}$, we have

$$\dim M - 1 \leq \dim M/xM \leq \dim M.$$

(for proof: see A–M or Eisenbud)

The max'l ideal of $R/(x_1, \dots, x_i)$ is the image of $(x_{i+1}, \dots, x_d)$, so the dimension is at most, and thus equal to $d-i$. Thus it is regular and thus an integral domain.

The image of x_{i+1} in $R/(x_1, \dots, x_i)$ is nonzero (by minimality of generating set), so it's a NZD. $\square$

Discrete valuation rings (DVRs)

Def: A DVR is a regular local ring $(R, \mathfrak{m})$ of dimension 1. If $t \in \mathfrak{m}$ s.t. $\mathfrak{m} = (t)$ (i.e. t is a s.o.p.), then t is called a uniformizing parameter for R.

(unique up to mult. by a unit)

Note: Regular local rings of dimension d correspond to "smooth" points $P \in \operatorname{Spec} R$, where $\operatorname{codim} P = d$. In the case $d=1$, P corresponds to a codimension one subscheme (or a closed point if $\operatorname{Spec} R$ is a curve), and R_P is a DVR.

Prop: Let R be a DVR, $t \in R$ a uniformizing parameter, and K the field of fractions of R. Then every nonzero $z \in K$ can be written uniquely as $z = ut^n$, where u is a unit in R, and $n \in \mathbb{Z}$

Pf: First we show uniqueness: if $ut^n = vt^m$ and $n \geq m$, then $ut^{n-m} = v$, so t^{n-m} is a unit in R, so $n = m$ and $u = v$.

Now, take $z \in K$. First we assume $z \in R$. Then if z is a unit in R, we're done. Otherwise $z \in (t)$, so $z = z_1 t$. If z_1 is a unit, we're done. Otherwise $z_1 = z_2 t$, and we get $z_1, z_2, \ldots$ with $z_n = z_{n+1} t$.

If some z_i is a unit, then $z = z_i t^i$, and we're done. Otherwise $(z_1) \subsetneq (z_2) \subsetneq \cdots$, which stabilizes by Noeth. $\Rightarrow$ $(z_n) = (z_{n+1})$ some n, so $z_{n+1} = v z_n$, some $v \in R$.

$\Rightarrow$ $z_n = z_{n+1} t = v z_n t$ $\Rightarrow$ $vt = 1$, a contradiction.

Now for arbitrary $z \in K$, write $z = \frac{r}{s}$, $r, s \in R$. Then $r = ut^m$, $s = vt^n$, so $z = \left(\frac{u}{v}\right) t^{m-n}$ and u/v is a unit in R. $\square$

Cor: The ideals of R are exactly $(1) \supsetneq (t) \supsetneq (t^2) \supsetneq \cdots$. In particular, R is a PID.

Ex: $\left(k[x,y]/(y-x^2)\right)_{(x,y)}$ is a DVR. Its max'l ideal is generated by $(x,y)=(x,x^2)=(x)$, so x is a uniformizing parameter.

Def: If R is a DVR with uniformizing parameter t and $K=K(R)$, the valuation corresponding to R is the function $v: K^* \to \mathbb{Z}$ defined

$$\underset{\substack{\uparrow \\ \text{unit}}}{z=ut^n} \longmapsto n.$$

Note that since any two uniformizing parameters differ by a unit, the valuation doesn't depend on choice of t.

If $z \in K$, then

$$v(z) \geq 0 \iff z \in R$$

$$v(z) = 0 \iff z \text{ a unit}$$

$$v(z) > 0 \iff z \in (t).$$

Moreover, if $v(z)>0$, then $v(z) = \max\{n \mid z \in (t^n)\}$, and if $v(z)<0$ then $v(1/z)>0$, so the above applies.

We can extend the valuation to K by defining $v(0)=\infty$, and get:

- $v(ab) = v(a) + v(b)$ (so $v : K^* \to \mathbb{Z}$ is a group homomorphism)

- $v(a+b) \geq \min\{v(a), v(b)\}$ (exercise).

Connections to normality

Let R be an integral domain. Recall that R is <u>normal</u> if it is its own integral closure in its field of fractions K.

We can check normality of R by checking that certain localizations are DVRs:

Theorem: R is normal $\iff$ for every prime $P \subseteq R$ associated to a principal ideal, R_P is principal.

We won't prove this, but recall that the associated primes of (f) include all primes minimal over (f) (and possibly more).

We know that for $P \neq 0$, $\operatorname{codim} P \geq 1$ since we're in an int. domain. The condition that P_P is principal thus says that $\operatorname{codim} P = 1$.

We also know that if codim $P=1$ then, P_p is principal $\Leftrightarrow$ R_p is a DVR. Thus, since every codim 1 prime is minimal over some principal ideal, we can restate the theorem as:

R is normal $\Longleftrightarrow$ all primes associated to principal ideals have codim 1 and R_p is a DVR for all codim 1 primes.

This leads to the following very useful corollary:

Cor: If R is normal then $R = \bigcap_{\substack{P \text{ prime,} \\ \text{codim } P = 1}} R_p$.

(Note that for any int. domain, $R = \bigcap_{P \text{ prime}} R_p$).

Pf: let $K := K(R)$, $a/u \in K$. If $a/u \notin R$, then a is not divisible by u. i.e. $a \notin (u)$.

Thus, $a \neq 0$ in $R/(u)$, so there is some (Associated!) prime P of (u) s.t. $a \neq 0$ in $(R/(u))_p$. i.e. $a \notin (u)_p$, so $a/u \notin R_p$.

Thus, if a/u is in R_p for every P associated to a principal ideal, then $a/u \in R$.

By the theorem, all such P will have P_p principal and thus $\operatorname{codim} P_p = \operatorname{codim} P = 1$. $\square$